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Bilinear forms on fermionic Novikov algebras

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Abstract

Novikov algebras were introduced in connection with the Poisson brackets of hydrodynamic type and Hamiltonian operators in formal variational calculus. Fermionic Novikov algebras correspond to a certain Hamiltonian super-operator in a super-variable. In this paper, we show that there is a remarkable geometry on fermionic Novikov algebras with non-degenerate invariant symmetric bilinear forms, which we call pseudo-Riemannian fermionic Novikov algebras. They are related to pseudo-Riemannian Lie algebras. Furthermore, we obtain a procedure to classify pseudo-Riemannian fermionic Novikov algebras. As an application, we give the classification in dimension $\leqslant 4$. Motivated by the one in dimension 4, we construct some examples in high dimensions.

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1. Introduction

Gel′fand and Dikii gave a bosonic formal variational calculus in [1, 2] and Xu gave a fermionic formal variational calculus in [3]. Moreover, motivated by the super-symmetric theory, a formal variational calculus of super-variables was given by Xu in [4] which combines the bosonic theory of Gel′fand–Dikii and the fermionic theory. Fermionic Novikov algebras are related to the Hamiltonian super-operator in terms of this theory. A fermionic Novikov algebra A is a vector space over a field $\mathbb{F}$ with a bilinear product $(x, y) \to xy$ satisfying

$$(xy)z - x(yz) = (yx)z - y(xz) \tag{1}$$

and

$$(xy)z = -(xz)y \tag{2}$$

for any $x, y, z \in A$. It corresponds to the following Hamiltonian operator H of type 0 [4]:

$$H^0_{\alpha,\beta} = \sum_{\gamma\in I} \left(a^{\gamma}_{\alpha,\beta}\Phi_{\gamma}(2) + b^{\gamma}_{\alpha,\beta}\Phi_{\gamma} D\right) \qquad a^{\gamma}_{\alpha,\beta}, b^{\gamma}_{\alpha,\beta} \in \mathbb{R}. \tag{3}$$

Fermionic Novikov algebras are a special class of left-symmetric algebras which only satisfy equation (1). Left-symmetric algebras are a class of non-associative algebras arising from the study of affine manifolds, affine structures and convex homogeneous cones [5–9]. A Novikov algebra was introduced as a left-symmetric algebra with commutative right multiplication operators: an algebra is a Novikov algebra if its product satisfies equation (1) and

$$(xy)z = (xz)y. \tag{4}$$

It connects with the Poisson brackets of hydrodynamic type [10–12] and Hamiltonian operators in the formal variational calculus [1–4, 13, 14]. The commutator of a left-symmetric A

$$[x, y] = xy - yx \tag{5}$$

defines a (sub-adjacent) Lie algebra $\mathfrak{g}(A)$.

There has been a lot of progress in the study of Novikov algebras [15–25]. However, we know very little about fermionic Novikov algebras except one real non-bosonic fermionic Novikov algebras of six dimensions [4], some non-bosonic fermionic Novikov algebras in low dimensions and some fermionic Novikov algebras in high dimensions [26].

A pseudo-Riemannian connection is a pseudo-metric connection such that the torsion is zero and parallel translation perseveres the bilinear form on the tangent spaces [27]. The corresponding structure on a fermionic Novikov algebra A is a non-degenerate symmetric bilinear form $f : A \times A \to \mathbb{F}$ such that

$$f(xy, z) + f(y, xz) = 0, \qquad \forall\, x, y, z \in A. \tag{6}$$

Such a fermionic Novikov algebra is called a pseudo-Riemannian fermionic Novikov algebra. In this paper, we show that the (sub-adjacent) Lie algebra of a pseudo-Riemannian fermionic Novikov algebra is a pseudo-Riemannian Lie algebra, which was first introduced in [28] and strongly related to pseudo-Riemannian Poisson manifolds [29] with compatible pseudo-metric.

The paper is organized as follows. In section 2, we show that the (sub-adjacent) Lie algebra of a pseudo-Riemannian fermionic Novikov algebra is the Lie algebra obtained by linearizing the Poisson structure at a point of a Poisson manifold with compatible pseudo-metric and a certain condition on the Levi-Civita contravariant connection. In section 3, we give a procedure to classify pseudo-Riemannian fermionic Novikov algebras. As an application, we list the classification in dimension $\leqslant 4$ in section 4. Motivated by the one in dimension 4, we construct some examples in high dimensions in section 5. In sections 6, we get some conclusions based on the discussion in the previous sections.

Throughout this paper we assume that the algebras are of finite dimension over $\mathbb{R}$.

2. Pseudo-Riemannian fermionic Novikov algebras

A pseudo-Riemannian fermionic Novikov algebra A is a fermionic Novikov algebra with a non-degenerate symmetric bilinear form f satisfying equation (6). Equations (1) and (2) are equivalent with equation (2) and

$$(xz)y - y(xz) + x(yz) - (yz)x = 0. \tag{7}$$

Therefore, the sub-adjacent Lie algebra $\mathfrak{g}(A)$ is a Lie algebra with a bilinear product $(x, y) \to xy$ satisfying equations (2) and (5) and

$$[xz, y] + [x, yz] = 0 \tag{8}$$

and a non-degenerate symmetric bilinear form f satisfying equation (6). It is a pseudo-Riemannian Lie algebra, which is a Lie algebra with a bilinear product $(x, y) \to xy$ satisfying equations (5) and (8) and a non-degenerate symmetric bilinear form f satisfying equation (6).

The notion of pseudo-Riemannian Lie algebras was first introduced in [28], which are strongly related to pseudo-Riemannian Poisson manifolds [29]. In fact, let P be a Poisson manifold with Poisson tensor π. A pseudo-metric of signature (p, q) on the cotangent bundle T^*P is a smooth symmetric contravariant 2-form $\langle\,,\,\rangle$ on P such that, at each point $x \in P$, $\langle\,,\,\rangle_x$ is non-degenerate on T^*_xP with signature (p, q). For any pseudo-metric $\langle\,,\,\rangle$ on T^*P, define a contravariant connection by

$$
\begin{aligned}
2\langle D_\alpha\beta, \gamma\rangle = {}& \sigma_\pi(\alpha)\cdot\langle\beta, \gamma\rangle + \sigma_\pi(\beta)\cdot\langle\alpha, \gamma\rangle - \sigma_\pi(\gamma)\cdot\langle\alpha, \beta\rangle \\
&+ \langle[\alpha, \beta]_\pi, \gamma\rangle + \langle[\gamma, \alpha]_\pi, \beta\rangle + \langle[\gamma, \beta]_\pi, \alpha\rangle
\end{aligned}
$$

where $\alpha, \beta, \gamma \in \Omega^1(P)$ and Lie bracket $[\,,\,]$ is given by

$$
\begin{aligned}
[\alpha, \beta]_\pi &= L_{\sigma_\pi(\alpha)}\beta - L_{\sigma_\pi(\beta)}\alpha - d(\pi(\alpha, \beta)) \\
&= i_{\sigma_\pi(\alpha)}d\beta - i_{\sigma_\pi(\beta)}d\alpha + d(\pi(\alpha, \beta)).
\end{aligned}
$$

Furthermore if

$$\pi(D_\alpha df, \beta) + \pi(\alpha, D_\beta df) = 0$$

for any $\alpha, \beta \in \Omega^1(P)$ and $f \in C^\alpha(P)$, then the triple $(P, \pi, \langle\,,\,\rangle)$ is called a pseudo-Riemannian Poisson manifold. When $\langle\,,\,\rangle$ is positive definite we call the triple a Riemann–Poisson manifold. Let f denote the restriction of $\langle\,,\,\rangle$ on $\mathrm{Ker}\sigma_\pi(x)$. Then, for any point $x \in P$ such that f is non-degenerate, the Lie algebra $\mathfrak{g}_x$ obtained by linearizing the Poisson structure at x is a pseudo-Riemannian Lie algebra. Let us enumerate some important applications of pseudo-Riemannian Lie algebras [28, 30].

(1) If $\mathfrak{g}$ is a pseudo-Riemannian Lie algebra, then there is a pseudo-metric $\langle\,,\,\rangle$ on the dual $\mathfrak{g}^*$ endowed with its linear Poisson structure π for which the triple $(\mathfrak{g}^*, \pi, \langle\,,\,\rangle)$ is a pseudo-Riemannian Poisson manifold.

(2) If $(P, \pi, \langle\,,\,\rangle)$ is a Riemann–Poisson manifold and S be a symplectic leaf of P, then S is a Kähler manifold.

(3) If $\mathfrak{g}$ is a Riemann–Lie algebra, then any even dimensional subalgebra of the orthogonal subalgebra defined in [30] gives rise to a structure of a Riemann–Poisson Lie group on any Lie group whose Lie algebra is $\mathfrak{g}$. Moreover, we get a structure of Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ where both $\mathfrak{g}$ and $\mathfrak{g}^*$ are Riemann–Lie algebras.

Claim 1. Furthermore, if the Levi-Civita contravariant connection D mentioned above satisfies

$$D_{D_\alpha\beta}\gamma = -D_{D_\alpha\gamma}\beta \tag{9}$$

for any $\alpha, \beta, \gamma \in \Omega^1(P)$, then $\mathfrak{g}_x$ is the sub-adjacent Lie algebra of a pseudo-Riemannian fermionic Novikov algebra.

3. Classification of pseudo-Riemannian fermionic Novikov algebras

Let $RZ(A) = \{x \in A \mid yx = 0, \forall\, y \in A\}$. Thus

$$RZ(A) = (AA)^\perp, \tag{10}$$

where $(AA)^\perp = \{x \in A \mid f(x, yz) = 0, \forall\, y, z \in A\}$.

In fact $\forall\, x, y, z \in A$,

$$x \in (AA)^{\perp} \Leftrightarrow f(yz, x) = 0 \Leftrightarrow f(z, yx) = 0 \Leftrightarrow yx = 0 \Leftrightarrow x \in RZ(A).$$

Definition 1. *$RZ(A)$ is called isotropic if $f(x, y) = 0, \forall\, x, y \in RZ(A)$, otherwise not isotropic.*

(1) If $RZ(A)$ is not isotropic, then there exists a non-degenerate ideal of A whose dimension equals to $\dim A - 1$.

In fact, since $RZ(A)$ is not isotropic, there exists an element of $RZ(A)$ such that $f(x, x) \neq 0$, which implies $A = \mathbb{F}x + x^{\perp}$ and $f\ |_{x^{\perp}\times x^{\perp}}$ is non-degenerate. Since $0 = f(zx, y) = -f(x, zy), \forall\, z \in A, y \in x^{\perp}$, then we have $zy \in x^{\perp}$, which implies $yz \in x^{\perp}$.

According to the above discussion, any pseudo-Riemannian fermionic Novikov algebra A with $RZ(A)$ not isotropic can be completely determined by a pseudo-Riemannian fermionic Novikov algebra whose dimension is $\dim A - 1$ as follows.

Let A_1 be any pseudo-Riemannian fermionic Novikov algebra with the bilinear form f_1 whose dimension is $\dim A - 1$ and A_2 be a pseudo-Riemannian fermionic Novikov algebra with the bilinear form f_2 in dimension 1. Define a new vector space by

$$A = A_1 + A_2. \tag{11}$$

Define a symmetric bilinear form f on A by

(1) $f|_{A_1\times A_1} = f_1$;
(2) $f|_{A_2\times A_2} = f_2$;
(3) $f|_{A_1\times A_2} = 0$.

Define a bilinear product $(u, v) \to uv$ on A by

(1) The product restricted on $A_i, i = 1, 2$, is respectively the product of A_i.
(2) $A_1A_2 = 0$.
(3) L_x is a derivation of A_1 for any $x \in A_2$.
(4) $(xy)y = 0, \forall\, x \in A_2, y \in A_1$.
(5) $f_1(xy, z) + f_1(y, xz) = 0$.

Claim 2. In terms of the product and bilinear form mentioned above, A is a pseudo-Riemannian fermionic Novikov algebra with $RZ(A)$ not isotropic. And any pseudo-Riemannian fermionic Novikov algebra with RZ not isotropic is obtainable in this manner.

(2) If $RZ(A)$ is isotropic, then

$$RZ(A) \subset (RZ(A))^{\perp} = AA \tag{12}$$

and

$$\dim RZ(A) \leqslant \frac{\dim A}{2}. \tag{13}$$

Furthermore, we have

$$1 \leqslant \dim RZ(A) \leqslant \frac{\dim A}{2}. \tag{14}$$

In fact, in terms of equation (10),

$$\dim RZ(A) + \dim AA = \dim A, \tag{15}$$

which implies

$$RZ(A) \neq 0 \tag{16}$$

since $AA \neq A$.

Choose a basis $\{e_1, \ldots, e_k, \ldots, e_n, \ldots, e_{n+k}\}$ of A such that $\{e_1, \ldots, e_k\}$ is a basis of $RZ(A)$, $\{e_1, \ldots, e_n\}$ is a basis of AA and

$$\begin{aligned}
&f(e_i, e_j) = \pm\delta_{ij}, && k+1 \leqslant i, j \leqslant n,\\
&f(e_i, e_{n+j}) = \delta_{ij}, && 1 \leqslant i, j \leqslant k,\\
&f(e_i, e_j) = 0, && 1 \leqslant i, j \leqslant k,\\
&f(e_i, e_j) = 0, && n+1 \leqslant i, j \leqslant n+k.
\end{aligned}$$

Thus, we can compute the structure constants under the above basis.

Based on the above discussion, we obtain a procedure to classify the pseudo-Riemannian fermionic Novikov algebras.

Step 1. Find pseudo-Riemannian fermionic Novikov algebras in dimension 1.

Step 2. Assume that we have got all pseudo-Riemannian fermionic Novikov algebras in dimension $p-1$ by induction.

Step 3. According to (1), compute pseudo-Riemannian fermionic Novikov algebras with RZ not isotropic in dimension p.

Step 4. Based on (2), compute pseudo-Riemannian fermionic Novikov algebras with RZ isotropic in dimension p for $\dim RZ = 1, 2, \ldots, \left[\frac{p}{2}\right]$, respectively.

In theory, we obtain all pseudo-Riemannian fermionic Novikov algebras. But the calculation is very difficult, especially for step 4, even in low dimensions. And there is another problem unsolved. That is, we probably get the same one induced from two different pseudo-Riemannian fermionic Novikov algebras in dimension $p-1$ by step 3. Thus, we must verify which are identical and which are different. It is also a very hard work. However, we get a new way to classify pseudo-Riemannian fermionic Novikov algebras.

4. Classification of pseudo-Riemannian fermionic Novikov algebras in dimension $\leqslant 4$

In the previous section, we give a procedure to classify the pseudo-Riemannian fermionic Novikov algebras in any dimension. As an application, we get pseudo-Riemannian fermionic Novikov algebras in dimension $\leqslant 4$.

Let $\{e_1, e_2, \ldots, e_n\}$ be a basis of A, then we have

$$f(e_ie_j, e_k) + f(e_j, e_ie_k) = 0. \tag{17}$$

Moreover, a bilinear form on A under the basis $\{e_1, e_2, \ldots, e_n\}$ is completely decided by the matric $F = (f_{ij})$, where

$$f_{ij} = f(e_i, e_j). \tag{18}$$

Let $\{c_{ij}^k\}$ be the set of structure constants of A, i.e.,

$$e_i e_j = \sum_k c_{ij}^k e_k. \tag{19}$$

Denote the (form) character matrix of a pseudo-Riemannian fermionic Novikov algebra by

$$\begin{pmatrix} \sum_k c_{11}^k e_k & \cdots & \sum_k c_{1n}^k e_k \\ \vdots & \ddots & \vdots \\ \sum_k c_{n1}^k e_k & \cdots & \sum_k c_{nn}^k e_k \end{pmatrix}.$$

Theorem 1. *The classification of pseudo-Riemannian fermionic Novikov algebras in dimension $\leqslant 2$ is given as follows:*

Character matrix	Dimension	Non-degenerate symmetric Bilinear form satisfying (6)	Notation
(1)(0)	1	$F = (a)$	$a = \pm 1$
(2) $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$	2	$F = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$	$a = \pm 1$, $b = \pm 1$.

Proof.

(1) It is trivial when $\dim A = 1$.

(2) $\dim A = 2$.

(a) $RZ(A)$ is isotropic. Then $\dim RZ(A) = 1$ and there exists a basis $\{e_1, e_2\}$ of A such that e_1 is a basis of $RZ(A) = AA$ and $f(e_1, e_2) = f(e_2, e_1) = 1$. Let $e_1e_2 = ae_1, e_2e_2 = be_1$. Then $a = 0$ since $f(e_1e_2, e_2) + f(e_2, e_1e_2) = 0$ and $b = 0$ since $f(e_2e_2, e_2) + f(e_2, e_2e_2) = 0$. It is a contradiction.

(b) $RZ(A)$ is not isotropic. There exists a basis $\{e_1, e_2\}$ of A such that $f(e_1, e_1) = \pm 1$, $f(e_2, e_2) = \pm 1$ and $e_1e_2 = ae_2$. Then $a = 0$ since $f(e_1e_2, e_2) + f(e_2, e_1e_2) = 0$. It is (2). $\square$

Theorem 2. *The classification of pseudo-Riemannian fermionic Novikov algebras in dimension 3 is given as follows:*

Character matrix	Non-degenerate symmetric Bilinear form satisfying (6)	Notation
$(T1)$ $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$F = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$	$a = \pm 1$, $b = \pm 1$, $c = \pm 1$
$(T2)$ $\begin{pmatrix} -\frac{1}{a}e_2 & e_3 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$F = \begin{pmatrix} 0 & 0 & 1 \\ 0 & a & 0 \\ 1 & 0 & 0 \end{pmatrix}$	$a \neq 0$
$(T3)$ $\begin{pmatrix} 0 & e_3 & ce_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$F = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & -bc \end{pmatrix}$	$a \neq 0$, $b \neq 0$, $c \neq 0$.

Theorem 3. *The classification of pseudo-Riemannian fermionic Novikov algebras in dimension 4 is given as follows:*

Character matrix	Non-degenerate symmetric Bilinear form satisfying (6)	Notation
$(F1)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{pmatrix}$	$F=\begin{pmatrix}a&0&0&0\\0&b&0&0\\0&0&c&0\\0&0&0&d\end{pmatrix}$	$a=\pm1$, $b=\pm1$, $c=\pm1$, $d=\pm1$
$(F2)\begin{pmatrix}-\frac{1}{a}e_2&e_3&0&0\\0&0&0&0\\0&0&0&0\\-\frac{k}{a}e_2&ke_3&0&0\end{pmatrix}$	$F=\begin{pmatrix}0&0&1&0\\0&a&0&0\\1&0&0&0\\0&0&0&b\end{pmatrix}$	$a\neq0$, $b=\pm1$
$(F3)\begin{pmatrix}0&e_3&ce_2&0\\0&0&0&0\\0&0&0&0\\0&ke_3&kce_2&0\end{pmatrix}$	$F=\begin{pmatrix}a&0&0&0\\0&b&0&0\\0&0&-bc&0\\0&0&0&d\end{pmatrix}$	$a\neq0$, $b\neq0$, $c\neq0$, $d=\pm1$
$(F4)\begin{pmatrix}0&0&0&0\\0&0&0&0\\0&0&e_2&-e_1\\0&0&0&0\end{pmatrix}$	$F=\begin{pmatrix}0&0&1&0\\0&0&0&1\\1&0&0&0\\0&1&0&0\end{pmatrix}$	

Here, we only sketch the proofs of theorems 2 and 3 since they are very long calculations and similar to theorem 1.

(1) $\dim A = 3$.

(a) $RZ(A)$ is isotropic. Then $\dim RZ(A) = 1$ and there exists a basis $\{e_1, e_2, e_3\}$ of A such that e_3 is a basis of $RZ(A)$ and $\{e_2, e_3\}$ is a basis of AA and $f(e_2, e_2) = a \neq 0$, $f(e_1, e_3) = f(e_3, e_1) = 1$. Calculating the structure constants, we have $e_1e_1 = -\frac{1}{a}e_2$, $e_1e_2 = e_3$, which is $(T2)$.

(b) $RZ(A)$ is not isotropic. Then A is induced from a pseudo-Riemannian fermionic Novikov algebra A_1 in dimension 2. Then there exists a basis $\{e_1, e_2, e_3\}$ of A such that $\{e_2, e_3\}$ is basis of A_1 and $f(e_1, e_1) = a \neq 0$, $f(e_2, e_2) = b \neq 0$, $f(e_3, e_3) = k \neq 0$. Since $f(e_1x, x) + f(x, e_1x) = 0$ for any $x \in A_1$, then $e_1e_2 = me_3$, $e_1e_3 = ne_2$. We have $mk + nb = 0$ since $f(e_1e_2, e_3) + f(e_2, e_1e_3) = 0$.

(i) If $m = n = 0$, we get $(T1)$.

(ii) $m \neq 0, n \neq 0$. Replacing e_3 by me_3 and taking $c = \frac{n}{m}$, we get $(T3)$.

(2) $\dim A = 4$.

(a) $RZ(A)$ is isotropic. Then $\dim RZ(A) = 1$ or 2. But it is impossible when $\dim RZ(A) = 1$. If $\dim RZ(A) = 2$, then there exists a basis $\{e_1, e_2, e_3, e_4\}$ of A such that $\{e_1, e_2\}$ is a basis of $RZ(A) = AA$ and $f(e_1, e_3) = f(e_2, e_4) = f(e_3, e_1) = f(e_4, e_2) = 1$. Calculating the structure constants, we have $e_3e_3 = e_2$, $e_3e_4 = -e_1$, which is $(F4)$.

(b) $RZ(A)$ is not isotropic. Then A is induced from a pseudo-Riemannian fermionic Novikov algebra A_1 in dimension 3.

(i) If A_1 is type $(T2)$, then there exists a basis $\{e_1, e_2, e_3, e_4\}$ of A such that $\{e_1, e_2, e_3\}$ is basis of A_1 and $f(e_4, e_4) = b$. It is not hard to get that

$$e_4e_1 = -\frac{k}{a}e_2, \qquad e_4e_2 = -ke_3, \qquad e_4e_3 = 0$$

by equations (6) and (7).

(ii) If A_1 is type $(T3)$, similar to the above case, we get

$$e_4e_1 = 0, \qquad e_4e_2 = ke_3, \qquad e_4e_3 = kce_2.$$

(iii) The simplest type induced from $(T1)$ is $(F1)$. The other ones are $(F2)$ and $(F3)$ with $k = 0$.

These are some remarks on these theorems.

(1) $(T2)$ is the only one with RZ isotropic in dimension 3.

(2) $(F4)$ is the only one with RZ isotropic in dimension 4 and $\dim RZ = 2$.

(3) There does not exist a pseudo-Riemannian fermionic Novikov algebra A with $RZ(A)$ isotropic and $\dim RZ(A) = 1$ in dimension 4.

(4) We can get $(F2)$ and $(F3)$ induced from $(T2)$ and $(T3)$, respectively. But induced from $(T1)$, we get $(F1)$, part of $(F2)$ and $(F3)$. Hence, it is important to recognize which of those obtained by different inductions are identical.

(5) Bai has given the classification of Novikov algebras with such bilinear forms in dimension $\leqslant 3$ in [31]. Our methods are different, but the results are identical.

5. Some examples in high dimensions

Definition 2. *If $A = A_1 \oplus A_2$, where $f(A_1, A_2) = 0$ and $A_i, i = 1, 2$, are non-degenerate ideals of A, we call A decomposable, otherwise indecomposable.*

The indecomposable pseudo-Riemannian fermionic Novikov algebras with RZ isotropic play a crucial role. $(T2)$ and $(F4)$ are such examples. Motivated by $(F4)$, we construct a class of examples in high dimensions, some of which are indecomposable ones with RZ isotropic.

Let A be a vector space with a basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ in dimension $2n$, where $n \geqslant 2$. Define a bilinear form f on A under the basis $\{e_1, e_2, \ldots, e_n\}$ by the matrix

$$F = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Define a bilinear product $(u, v) \rightarrow uv$ on A satisfying

$$f_1 f_i = e_{n+1-i}, \qquad \forall\, 1 \leqslant i \leqslant \left[\frac{n}{2}\right] \tag{20}$$

and

$$f_1 f_i = -e_{n+1-i}, \qquad \forall \left[\frac{n+1}{2}\right] + 1 \leqslant i \leqslant n \tag{21}$$

and otherwise zero.

If $n = 2k$, A is an indecomposable pseudo-Riemannian fermionic Novikov algebra with $RZ(A)$ spanned by $\{e_1, \ldots, e_n\}$ isotropic.

If $n = 2k + 1$, A is decomposable. In fact, let A_1 be the subspace of A spanned by $\{e_1, \ldots, \widehat{e}_{k+1}, \cdots, e_n, f_1, \ldots, \widehat{f}_{k+1}, \ldots, f_n\}$. Then A_1 is a non-degenerate ideal of A with $RZ(A_1)$ spanned $\{e_1, \ldots, \widehat{e}_{k+1}, \ldots, e_n\}$ isotropic. Moreover, A_1 is indecomposable. Let A_2

be the subspace of A spanned by $\{e_{k+1}, f_{k+1}\}$. Then A_2 is also a non-degenerate ideal of A with $RZ(A_2) = A_2$. Thus, we have $A = A_1 \oplus A_2$ and $RZ(A)$ spanned by $\{e_1, \ldots, e_n, f_{k+1}\}$ is not isotropic.

Here, we construct a class of indecomposable pseudo-Riemannian fermionic Novikov algebras with RZ isotropic in dimension $4k$. Hence, although there are not many pseudo-Riemannian fermionic Novikov algebras in dimension $\leqslant 4$, we believe that there exist many examples with RZ isotropic, therefore many in high dimensions.

6. Conclusion and discussion

According to the discussion in the previous sections, we obtain some conclusions on pseudo-Riemannian fermionic Novikov algebras and pseudo-Riemannian Lie algebras.

(1) The sub-adjacent of a pseudo-Riemannian fermionic Novikov algebra is the Lie algebra obtained by linearizing the Poisson structure at a point of a Poisson manifold with compatible pseudo-metric and a certain condition (9) on the Levi-Civita contravariant connection.
(2) The sub-adjacent Lie algebras of pseudo-Riemannian fermionic Novikov algebras are equivalent with pseudo-Riemannian Lie algebras in less than or equal to four dimensions.
(3) For a fermionic Novikov algebra A, $AA \neq A$. But for a pseudo-Riemannian Lie algebra $\mathfrak{g}$, we neither could prove $\mathfrak{g} \neq \mathfrak{g}\mathfrak{g}$ nor found an example with $\mathfrak{g} = \mathfrak{g}\mathfrak{g}$.

Although there are no answers for some questions, it is useful for getting such connections among algebra structure, geometry and physics.

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